Problem 1.16

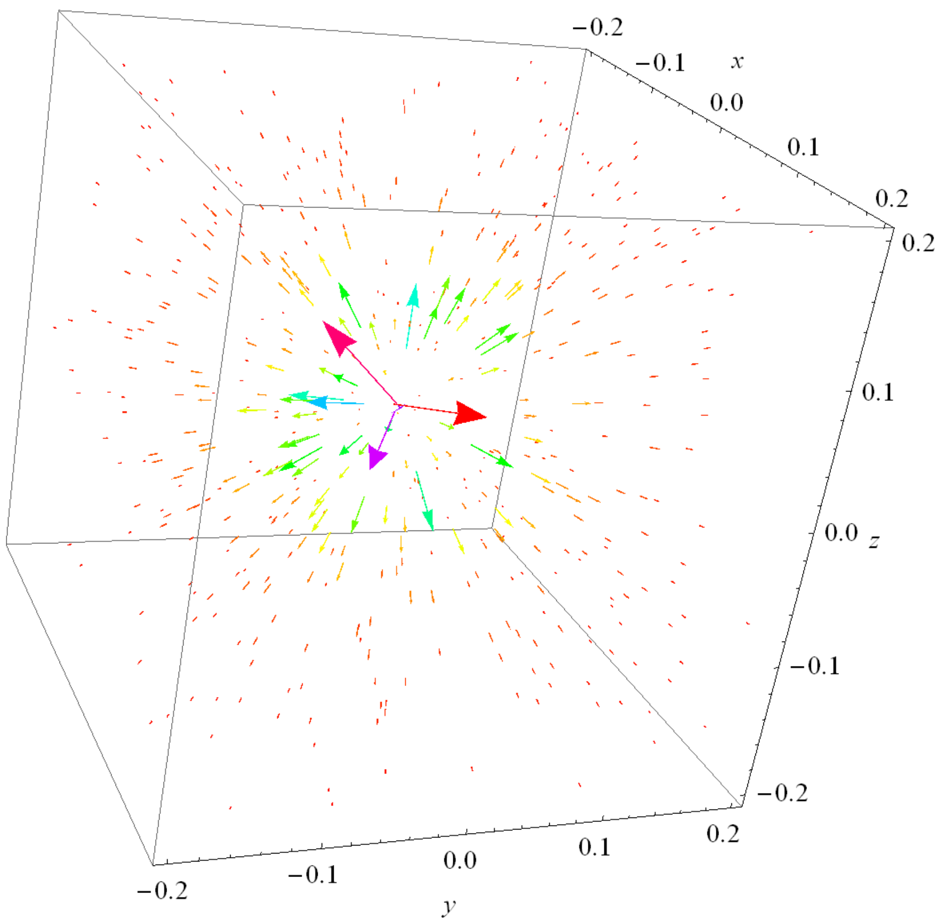
Sketch the vector function

$$\mathbf{v} = \frac{\hat{\mathbf{r}}}{r^2},$$

and compute its divergence. The answer may surprise you ... how do you account for it?

Solution

Below is a sketch of the vector field in a small box. The hue and length of a vector indicate its magnitude.



To compute the divergence of this completely radial vector function, it's easiest to use the formula in spherical coordinates inside the front cover, plugging in $v_r = 1/r^2$, $v_\theta = 0$, and $v_\phi = 0$. This hasn't been developed yet in the textbook, so it will be done in Cartesian coordinates. Rewrite the vector function.

$$\begin{aligned}\mathbf{v} = \frac{\hat{\mathbf{r}}}{r^2} = \frac{\mathbf{r}}{r^3} &= \frac{x\hat{\mathbf{x}} + y\hat{\mathbf{y}} + z\hat{\mathbf{z}}}{(x^2+y^2+z^2)^{3/2}} \\ &= \frac{x}{(x^2+y^2+z^2)^{3/2}}\hat{\mathbf{x}} + \frac{y}{(x^2+y^2+z^2)^{3/2}}\hat{\mathbf{y}} + \frac{z}{(x^2+y^2+z^2)^{3/2}}\hat{\mathbf{z}}\end{aligned}$$

And then take the divergence.

$$\begin{aligned}
\nabla \cdot \mathbf{v} &= \frac{\partial}{\partial x}\left[\frac{x}{(x^2+y^2+z^2)^{3/2}}\right] \\
&\quad + \frac{\partial}{\partial y}\left[\frac{y}{(x^2+y^2+z^2)^{3/2}}\right] \\
&\qquad + \frac{\partial}{\partial z}\left[\frac{z}{(x^2+y^2+z^2)^{3/2}}\right] \\
&= \left[\frac{1}{(x^2+y^2+z^2)^{3/2}} - \frac{3}{2}\frac{x}{(x^2+y^2+z^2)^{5/2}}\cdot 2x\right] \\
&\quad + \left[\frac{1}{(x^2+y^2+z^2)^{3/2}} - \frac{3}{2}\frac{y}{(x^2+y^2+z^2)^{5/2}}\cdot 2y\right] \\
&\qquad + \left[\frac{1}{(x^2+y^2+z^2)^{3/2}} - \frac{3}{2}\frac{z}{(x^2+y^2+z^2)^{5/2}}\cdot 2z\right] \\
&= \frac{3}{(x^2+y^2+z^2)^{3/2}} - 3\frac{x^2+y^2+z^2}{(x^2+y^2+z^2)^{5/2}} \\
&= \frac{3}{(x^2+y^2+z^2)^{3/2}} - \frac{3}{(x^2+y^2+z^2)^{3/2}} \\
&= 0
\end{aligned}$$

This argument assumes implicitly that $x^2 + y^2 + z^2 \neq 0$; that is, away from the origin the divergence of this vector field is zero. See part (b) of Problem 1.39 for the full story.